

PARISIAN RUIN OF BROWNIAN MOTION RISK MODEL OVER AN INFINITE-TIME HORIZON

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Abstract: Let $B(t), t \in \mathbb{R}$ be a standard Brownian motion. In this paper, we derive the exact asymptotics of the probability of Parisian ruin on infinite time horizon for the following risk process

$$(0.1) \quad R_u^\delta(t) = e^{\delta t} \left(u + c \int_0^t e^{-\delta v} dv - \sigma \int_0^t e^{-\delta v} dB(v) \right), \quad t \geq 0,$$

where $u \geq 0$ is the initial reserve, $\delta \geq 0$ is the force of interest, $c > 0$ is the rate of premium and $\sigma > 0$ is a volatility factor. Further, we show the asymptotics of the Parisian ruin time of this risk process.

Key Words: Parisian ruin; ruin probability; ruin time; Brownian motion

AMS Classification: Primary 60G15; secondary 60G70

1. INTRODUCTION

In the risk theory, the surplus process of an insurance company can be modeled by

$$R_u(t) = u + ct - X(t), \quad t \geq 0,$$

see [11], where $u \geq 0$ is the initial reserve, ct models the total premium received up to time t , and $X(t), t \geq 0$ denotes the aggregate claims process. In [7, 8], the Parisian ruin of $R_u(t)$ is defined by

$$(1.1) \quad \mathcal{P}_S(u, T_u) = \mathbb{P} \left\{ \inf_{t \in [0, S]} \sup_{s \in [t, t+T_u]} R_u(s) < 0 \right\}, \quad S \in (0, \infty],$$

where T_u models the pre-specified time which is a function of u . For $X(t), t \geq 0$ a Gaussian process, the asymptotics of $\mathcal{P}_S(u, T_u)$ over finite-time horizon, i.e. $S \in (0, \infty)$, is investigated in [8]. Further, [7] showed the tail asymptotic results of $R_u(t)$ over infinite-time horizon, i.e. $S = \infty$ in (1.1), where $X(t)$ is a self-similar Gaussian process. In this paper considering the nature of the financial market, we introduce the force of interest δ into the model $R_u(t)$ as $R_u^\delta(t)$ in (0.1) when $X(t) = B(t)$. [4] gave an approximation of the Parisian ruin probability

$$\mathcal{K}_S^\delta(u, T_u) := \mathbb{P} \left\{ \inf_{t \in [0, S]} \sup_{s \in [t, t+T_u]} R_u^\delta(s) < 0 \right\}, \quad S \in (0, \infty),$$

as $u \rightarrow \infty$. See [19, 6, 15] for more studies on risk models with force of interest. In the literature, no results are available for the approximation of Parisian ruin probability over infinite time horizon for $\delta > 0$. In this contribution we shall investigate the asymptotics of the Parisian ruin probability

$$\mathcal{K}^\delta(u, T_u) := \mathbb{P} \left\{ \inf_{t \geq 0} \sup_{s \in [t, t+T_u]} R_u^\delta(s) < 0 \right\},$$

as $u \rightarrow \infty$ where $T_u \geq 0$ models the pre-specified time satisfying

$$(1.2) \quad \lim_{u \rightarrow \infty} T_u = T \in [0, \infty].$$

When $\delta = 0$ and $T \in [0, \infty)$, [7] showed that (hereafter \sim means asymptotic equivalence)

$$\mathcal{K}^0(u, T_u) = \mathbb{P} \left\{ \inf_{t \geq 0} \sup_{s \in [t, t+T_u]} (u + cs - \sigma B(s)) < 0 \right\} \sim \mathcal{F} \left(\frac{2c^2 T}{\sigma^2} \right) \exp \left(-\frac{2cu}{\sigma^2} \right), \quad u \rightarrow \infty,$$

where

$$\mathcal{F}(T) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, T]} e^{\sqrt{2}B(t+s) - (t+s)} \right\}.$$

Hereafter we make the convention that $\sup \{\emptyset\} = 0$ and $\inf \{\emptyset\} = \infty$.

Complementary, we investigate the conditional distribution of the ruin time for the surplus process $R_u^\delta(t)$. The classical ruin time, e.g., [6, 13, 16], is defined as

$$(1.3) \quad \tau(u) = \inf\{t > 0 : R_u^\delta(t) < 0\}.$$

Here as in [7, 4] we define the Parisian ruin time of the risk process $R_u^\delta(t)$ by

$$(1.4) \quad \eta(u) = \inf\{t \geq T_u : t - \kappa_{t,u} \geq T_u, R_u^\delta(t) < 0\}, \quad \text{with } \kappa_{t,u} = \sup\{s \in [0, t] : R_u^\delta(s) \geq 0\},$$

and $\tau(u) = \eta(u)$ when $T_u \equiv 0$.

Brief outline of the rest of the paper: In Section 2 we present our main results on the asymptotics of $\mathcal{K}^\delta(u, T_u)$ as $u \rightarrow \infty$ and the approximation of the Parisian ruin time. All the proofs are relegated to Section 3.

2. MAIN RESULTS

Before giving the main results, we shall introduce a constant as

$$(2.1) \quad \tilde{\mathcal{P}}_a^f[0, \infty) = \lim_{\lambda \rightarrow \infty} \tilde{\mathcal{P}}_a^f[0, \lambda] \in (0, \infty),$$

with

$$\tilde{\mathcal{P}}_a^f[0, \lambda] = \mathbb{E} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [a, 1]} \exp \left(\sqrt{2}B(st) - st - f(st) \right) \right\} \in (0, \infty),$$

where $\lambda \geq 0, a \in [0, 1]$ and $f(t)$ is a continuous function satisfying $\lim_{t \rightarrow \infty} \frac{f(t)}{t^\epsilon} = \infty$ for some $\epsilon > 0$.

Note further that $\tilde{\mathcal{P}}_0^f[0, \lambda] = e^{-f(0)}$ and

$$\tilde{\mathcal{P}}_1^f[0, \lambda] = \mathbb{E} \left\{ \sup_{t \in [0, \lambda]} \exp \left(\sqrt{2}B(t) - |t| - f(t) \right) \right\},$$

see e.g. [9, 3, 14] for the bounds of $\tilde{\mathcal{P}}_a^f[0, \infty)$ and more details.

Recall that $\Phi(\cdot), \Psi(\cdot)$ denote the distribution function and the survival function of an $\mathcal{N}(0, 1)$ random variable, respectively, and $\Psi(u) \sim \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}, u \rightarrow \infty$.

Theorem 2.1. *For $\delta > 0$ and T_u satisfying (1.2), we have*

$$\begin{aligned} \mathcal{K}^\delta(u, T_u) &\sim \mathbb{E} \left\{ \sup_{t \in [0, \infty)} \inf_{s \in [a, 1]} \exp \left(\sqrt{2}B(st) - st - \left(\sqrt{st} - \frac{c}{\sigma\sqrt{\delta}} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right) \\ &= \tilde{\mathcal{P}}_a^f[0, \infty) \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right), \quad u \rightarrow \infty, \end{aligned}$$

where $a = e^{-2\delta T}$ and $f(t) = \left(\sqrt{t} - \frac{c}{\sigma\sqrt{\delta}} \right)^2$.

Remark 2.2. *In Theorem 2.1, if $T = 0, a = 1$, we get the asymptotic result of the classical ruin probability, i.e., as $u \rightarrow \infty$*

$$\mathcal{K}^\delta(u, 0) = \mathbb{P} \left\{ \inf_{t \geq 0} R_u^\delta(s) < 0 \right\} \sim \mathbb{E} \left\{ \sup_{t \in [0, \infty)} \exp \left(\sqrt{2}B(t) - t - \left(\sqrt{t} - \frac{c}{\sigma\sqrt{\delta}} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right)$$

which corresponds to the results in [2].

Moreover, according to [12] (see also [10]) we have

$$(2.2) \quad \mathcal{K}^\delta(u, 0) = \Psi\left(\frac{\sqrt{2\delta}}{\sigma}\left(u + \frac{c}{\delta}\right)\right) / \Psi\left(\frac{\sqrt{2c}}{\sigma\sqrt{\delta}}\right).$$

Theorem 2.3. Let $\eta(u)$ satisfy (1.4), under the assumptions and notation of Theorem 2.1, we have for $\delta > 0$ and $x \in \left(-\frac{c^2}{\delta^2}, \infty\right)$

$$(2.3) \quad \mathbb{P}\left\{u^2\left(e^{-2\delta\eta(u)} - \left(\frac{c}{\delta u + c}\right)^2\right) \leq x \mid \eta(u) < \infty\right\} \sim \frac{\tilde{\mathcal{P}}_a^f[0, \frac{c^2}{\sigma^2\delta} + \frac{\delta x}{\sigma^2}]}{\tilde{\mathcal{P}}_a^f[0, \infty)}, \quad u \rightarrow \infty.$$

Remarks 2.4. i) When $\delta = 0$, [7] showed that for $x \in \mathbb{R}$

$$\mathbb{P}\left\{u^{-\frac{1}{2}}\left(\eta(u) - \frac{u}{c}\right) \leq x \mid \eta(u) < \infty\right\} \sim \Phi(cx), \quad u \rightarrow \infty.$$

ii) When $T_u \equiv 0$, $\eta(u) = \tau(u)$, by (2.3), we have

$$\mathbb{P}\left\{u^2\left(e^{-2\delta\tau(u)} - \left(\frac{c}{\delta u + c}\right)^2\right) \leq x \mid \eta(u) < \infty\right\} \sim \frac{\tilde{\mathcal{P}}_1^f[0, \frac{c^2}{\sigma^2\delta} + \frac{\delta x}{\sigma^2}]}{\tilde{\mathcal{P}}_1^f[0, \infty)}, \quad u \rightarrow \infty,$$

which corresponds to the result in [2].

3. PROOFS

Hereafter we assume that $\mathbb{C}_i, i \in \mathbb{N}$ are positive constants.

PROOF OF THEOREM 2.1 We have for $u > 0$

$$\mathcal{K}^\delta(u, T_u) = \mathbb{P}\left\{\inf_{t \in [0, \infty)} \sup_{s \in [t, t+T_u]} R_u^\delta(s) < 0\right\} = \mathbb{P}\left\{\inf_{t \in [0, \infty)} \sup_{s \in [t, t+T_u]} \tilde{R}_u^\delta(s) < 0\right\},$$

where

$$\tilde{R}_u^\delta(s) = u + c \int_0^s e^{-\delta v} dv - \sigma \int_0^s e^{-\delta v} dB(v), \quad t \geq 0.$$

Since for $t \in (0, \infty)$

$$\mathbb{E}\left\{\left[\sigma \int_0^t e^{-\delta v} dB(v)\right]^2\right\} = \frac{\sigma^2}{2\delta} (1 - e^{-2\delta t}),$$

then

$$\sup_{t \in [0, \infty)} \mathbb{E}\left\{\left[\sigma \int_0^t e^{-\delta v} dB(v)\right]^2\right\} < \infty$$

implies that

$$\sup_{t \in [0, \infty)} \mathbb{E}\left\{\left|\sigma \int_0^t e^{-\delta v} dB(v)\right|\right\} < \infty,$$

by the martingale convergence theorem, see [17], $\tilde{R}_u^\delta(\infty) := \lim_{t \rightarrow \infty} \tilde{R}_u^\delta(t)$ exists and is finite almost surely. Thus for any $u > 0$

$$\begin{aligned} \psi(u) &:= \mathbb{P}\left\{\inf_{t \in [0, \infty)} \sup_{s \in [t, t+T_u]} \tilde{R}_u^\delta(s) < 0\right\} = \mathbb{P}\left\{\inf_{t \in [0, \infty)} \sup_{s \in [t, t+T_u]} \tilde{R}_u^\delta(s) < 0\right\} \\ &= \mathbb{P}\left\{\sup_{t \in [0, \infty)} \inf_{s \in [t, t+T_u]} \left(\sigma \int_0^s e^{-\delta v} dB(v) - c \int_0^s e^{-\delta v} dv\right) > u\right\}. \end{aligned}$$

Using a change of variable $s = -\frac{1}{2\delta} \ln s^*$, $s^* \in [t^* e^{-2\delta T_u}, t^*]$, $t^* \in [0, 1]$, we have

$$\psi(u) = \mathbb{P}\left\{\sup_{t^* \in [0, 1]} \inf_{s^* \in [t^* e^{-2\delta T_u}, t^*]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dB(v) - c \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dv\right) > u\right\}$$

$$= \mathbb{P} \left\{ \sup_{t^* \in [0,1]} \inf_{s^* \in [t^* e^{-2\delta T_u}, t^*]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{*\frac{1}{2}}) \right) > u \right\}.$$

For simplicity, we still use s, t instead of s^*, t^* .

Below, we set $Z(s) = \sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v)$ with variance function given by

$$V_Z^2(s) = \text{Var} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) \right) = \frac{\sigma^2}{2\delta} (1 - s), \quad s \in [0, 1].$$

We show next that for u sufficiently large

$$M_u(t) := \frac{u V_Z(t)}{G_u(t)} = \frac{\frac{\sigma}{\sqrt{2\delta}} \sqrt{1-t}}{1 + \frac{c}{\delta u} (1 - t^{1/2})}, \quad 0 \leq t \leq 1,$$

with $G_u(t) := u + \frac{c}{\delta} (1 - t^{1/2})$ attains its maximum at the unique point

$$t_u = \left(\frac{c}{\delta u + c} \right)^2.$$

In fact, we have for $t \in (0, 1)$

$$\begin{aligned} [M_u(t)]_t &:= \frac{dM_u(t)}{dt} = \frac{dV_Z(t)}{dt} \cdot \frac{u}{G_u(t)} - \frac{V_Z(t)}{G_u^2(t)} \left(-\frac{cu}{2\delta} t^{-\frac{1}{2}} \right) \\ &= \frac{u}{2G_u^2(t)V_Z(t)} \left[\frac{dV_Z^2(t)}{dt} G_u(t) + V_Z^2(t) \frac{ct^{-\frac{1}{2}}}{\delta} \right] \\ (3.1) \quad &= \frac{u\sigma^2 t^{-1/2}}{4\delta G_u^2(t)V_Z(t)} \left[\frac{c}{\delta} - \left(u + \frac{c}{\delta} \right) t^{\frac{1}{2}} \right]. \end{aligned}$$

Letting $[M_u(t)]_t = 0$, we get $t_u = \left(\frac{c}{\delta u + c} \right)^2$.

By (3.1), $[M_u(t)]_t > 0$ for $t \in (0, t_u)$ and $[M_u(t)]_t < 0$ for $t \in (t_u, 1)$, so t_u is the unique maximum point of $M_u(t)$ over $[0, 1]$. Further

$$M_u := M_u(t_u) = \frac{\sigma u}{\sqrt{2\delta u^2 + 4cu}} = \frac{\sigma}{\sqrt{2\delta}} (1 + o(1)), \quad u \rightarrow \infty.$$

Set $\delta(u) = \left(\frac{\ln u}{u} \right)^2$, $\Delta(u) = [0, t_u + \delta(u)]$ and for some positive constant λ

$$I_u(k) = [k\lambda u^{-2}, (k+1)\lambda u^{-2}], \quad k \in \mathbb{N}, \quad N(u) = \lfloor \lambda^{-1} (\ln u)^2 \rfloor.$$

We have for u large enough

$$(3.2) \quad \psi(u) \geq \mathbb{P} \left\{ \sup_{t \in [0, t_u + \lambda u^{-2}]} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\} =: \Pi_0(u),$$

$$(3.3) \quad \psi(u) \leq \Pi_0(u) + \Pi_1(u) + \Pi_2(u) + \Pi_3(u),$$

where for $\theta \in (0, 1)$

$$\begin{aligned} \Pi_1(u) &= \sum_{i=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in (t_u + I_u(k))} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\}, \\ \Pi_2(u) &= \mathbb{P} \left\{ \sup_{t \in [t_u + \delta(u), \theta]} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\}, \\ \Pi_3(u) &= \mathbb{P} \left\{ \sup_{t \in [\theta, 1]} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\}. \end{aligned}$$

First we show the asymptotic of $\Pi_0(u)$. For u large enough

$$\begin{aligned}
\Pi_0(u) &= \mathbb{P} \left\{ \sup_{t \in [0, t_u + \lambda u^{-2}]} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\} \\
&= \mathbb{P} \left\{ \sup_{t \in [0, t_u + \lambda u^{-2}]} \inf_{s \in [te^{-2\delta T_u}, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\} \\
&\leq \mathbb{P} \left\{ \sup_{t \in [0, (1+\varepsilon_1)(c^2/\delta^2 + \lambda)u^{-2}]} \inf_{s \in [t(1+\varepsilon_2)a, t]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - s^{\frac{1}{2}}) \right) > u \right\} \\
&= \mathbb{P} \left\{ \sup_{t \in [0, (1+\varepsilon_1)(c^2/\delta^2 + \lambda)]} \inf_{s \in [(1+\varepsilon_2)a, 1]} \bar{Z}(stu^{-2}) \frac{M_u(stu^{-2})}{M_u} > \frac{u}{M_u} \right\} \\
&=: \Pi_0^{+\varepsilon}(u),
\end{aligned}$$

where $\bar{Z}(t) = \frac{Z(t)}{V_Z(t)}$, $a = e^{-2\delta T}$, $\varepsilon_1 \in (0, 1)$ and $\varepsilon_2 \in (0, (\frac{1}{a} - 1) \wedge 1)$ if $T \in (0, \infty]$, $\varepsilon_2 = 0$ if $T = 0$. Similarly,

$$\Pi_0(u) \geq \mathbb{P} \left\{ \sup_{t \in [0, (1-\varepsilon_1)(c^2/\delta^2 + \lambda)]} \inf_{s \in [(1-\varepsilon_2)a, 1]} \bar{Z}(stu^{-2}) \frac{M_u(stu^{-2})}{M_u} > \frac{u}{M_u} \right\} =: \Pi_0^{-\varepsilon}(u).$$

We have

$$1 - \frac{M_u(t)}{M_u} = \frac{[G_u(t)V_Z(t_u)]^2 - [G_u(t_u)V_Z(t)]^2}{G_u(t)V_Z(t_u)[V_Z(t)G_u(t_u) + G_u(t)V_Z(t_u)]}.$$

and

$$\begin{aligned}
[G_u(t)V_Z(t_u)]^2 - [G_u(t_u)V_Z(t)]^2 &= \left[\left(u + \frac{c}{\delta} \right) - \frac{c}{\delta} \sqrt{t} \right]^2 \frac{\sigma^2}{2\delta} (1 - t_u) - \left[\left(u + \frac{c}{\delta} \right) - \frac{c}{\delta} \sqrt{t_u} \right]^2 \frac{\sigma^2}{2\delta} (1 - t) \\
&= \left(u + \frac{c}{\delta} \right)^2 \frac{\sigma^2}{2\delta} (t - t_u) - 2 \left(u + \frac{c}{\delta} \right) \frac{c\sigma^2}{2\delta^2} (\sqrt{t} - \sqrt{t_u}) (1 - t_u) - \frac{c^2\sigma^2}{2\delta^3} (t - t_u) \\
&= \frac{\sigma^2}{2\delta} \left[\left(u + \frac{c}{\delta} \right)^2 - \left(\frac{c}{\delta} \right)^2 \right] (\sqrt{t} - \sqrt{t_u})^2 \\
&= \frac{\sigma^2}{2\delta} \left(u^2 + \frac{2c}{\delta} u \right) (\sqrt{t} - \sqrt{t_u})^2.
\end{aligned}$$

Since for any $t \in \Delta(u)$

$$\sqrt{\frac{\sigma^2}{2\delta} (1 - t_u - \delta(u))} \leq V_Z(t) \leq \sqrt{\frac{\sigma^2}{2\delta}}, \quad u + \frac{c}{\delta} - \frac{c}{\delta} \sqrt{t_u + \delta(u)} \leq G_u(t) \leq u + \frac{c}{\delta},$$

then for all large u

$$V_Z(t_u)G_u(t)[G_u(t)V_Z(t_u) + V_Z(t)G_u(t_u)] \leq \frac{\sigma^2}{\delta} \left(u + \frac{c}{\delta} \right)^2$$

and

$$\begin{aligned}
V_Z(t_u)G_u(t)[G_u(t)V_Z(t_u) + V_Z(t)G_u(t_u)] &\geq \frac{\sigma^2}{\delta} (1 - t_u - \delta(u)) \left(u + \frac{c}{\delta} - \frac{c}{\delta} \sqrt{t_u + \delta(u)} \right)^2 \\
&\geq \frac{\sigma^2}{\delta} \left[\left(u + \frac{c}{\delta} \right)^2 - u \right].
\end{aligned}$$

Consequently, we have

$$(3.4) \quad \lim_{u \rightarrow \infty} \sup_{t \in (u^2 \Delta(u))} \left| \left(1 - \frac{M_u(stu^{-2})}{M_u} \right) u^2 - \frac{1}{2} \left(\sqrt{ts} - \frac{c}{\delta} \right)^2 \right| = 0.$$

For $0 \leq t' \leq t < 1$, the correlation function of $Z(t)$ equals

$$\begin{aligned}
 r(t, t') &= \frac{\mathbb{E} \left\{ (\sigma \int_0^{-\frac{1}{2\delta} \ln t} e^{-\delta v} dB(v)) (\sigma \int_0^{-\frac{1}{2\delta} \ln t'} e^{-\delta v} dB(v)) \right\}}{\sqrt{\frac{\sigma^2}{2\delta} (1-t)} \sqrt{\frac{\sigma^2}{2\delta} (1-t')}} \\
 (3.5) \quad &= \frac{\sqrt{1-t}}{\sqrt{1-t'}} = 1 - \frac{t-t'}{\sqrt{1-t'}(\sqrt{1-t'} + \sqrt{1-t})},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \sup_{t, t' \in \Delta(u), t' \neq t} \left| \frac{1-r(t, t')}{\frac{1}{2}|t-t'|} - 1 \right| &= \sup_{t, t' \in \Delta(u), t' \neq t} \left| \frac{2}{\sqrt{1-t}(\sqrt{1-t'} + \sqrt{1-t})} - 1 \right| \\
 &\leq \frac{1}{1 - (\frac{c}{c+\delta u})^2 - (\frac{\ln u}{u})^2} - 1 \\
 (3.6) \quad &\rightarrow 0, \quad u \rightarrow \infty.
 \end{aligned}$$

For $t, t' \in [0, (1+\varepsilon_1)(\frac{c^2}{\delta^2} + \lambda)]$ and $s, s' \in (0, 1]$

$$\begin{aligned}
 &u^2 \text{Var} \left(\overline{Z}(stu^{-2}) \frac{M_u(stu^{-2})}{M_u} - \overline{Z}(s't'u^{-2}) \frac{M_u(s't'u^{-2})}{M_u} \right) \\
 &= \frac{u^2}{M_u^2} \mathbb{E} \left\{ \frac{Z(stu^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} - \frac{Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{s't'u^{-2}})} \right\}^2 \\
 &= \frac{u^2}{M_u^2} \mathbb{E} \left\{ \frac{Z(stu^{-2}) - Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} + \frac{Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} - \frac{Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{s't'u^{-2}})} \right\}^2 \\
 &= \frac{u^2}{M_u^2} (J_1(u) + J_2(u) + J_3(u)),
 \end{aligned}$$

where

$$\begin{aligned}
 J_1(u) &= \mathbb{E} \left\{ \left(\frac{Z(stu^{-2}) - Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} \right)^2 \right\}, \\
 J_2(u) &= 2 \left(\frac{1}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} - \frac{1}{1 + \frac{c}{\delta u}(1 - \sqrt{s't'u^{-2}})} \right) \mathbb{E} \left\{ \frac{(Z(stu^{-2}) - Z(s't'u^{-2}))Z(s't'u^{-2})}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} \right\} = 0, \\
 J_3(u) &= \left(\frac{1}{1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})} - \frac{1}{1 + \frac{c}{\delta u}(1 - \sqrt{s't'u^{-2}})} \right)^2 \mathbb{E} \left\{ (Z(s't'u^{-2}))^2 \right\}.
 \end{aligned}$$

Since for $t, t' \in [0, (1+\varepsilon_1)(\frac{c^2}{\delta^2} + \lambda)]$ and $s, s' \in (0, 1]$

$$\begin{aligned}
 \lim_{u \rightarrow \infty} \frac{u^2}{M_u^2} J_1(u) &= \lim_{u \rightarrow \infty} \frac{u^2}{M_u^2 (1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}}))^2} \mathbb{E} \left\{ (Z(stu^{-2}) - Z(s't'u^{-2}))^2 \right\} = |st - s't'|, \\
 \lim_{u \rightarrow \infty} \frac{u^2}{M_u^2} J_3(u) &= \lim_{u \rightarrow \infty} \frac{\sigma^2 (1 - s't'u^{-2}) u^2}{2\delta M_u^2} \left(\frac{\frac{c}{\delta u} (\sqrt{stu^{-2}} - \sqrt{s't'u^{-2}})}{\left(1 + \frac{c}{\delta u}(1 - \sqrt{stu^{-2}})\right) \left(1 + \frac{c}{\delta u}(1 - \sqrt{s't'u^{-2}})\right)} \right)^2 = 0,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 \lim_{u \rightarrow \infty} u^2 \text{Var} \left(\overline{Z}(stu^{-2}) \frac{M_u(stu^{-2})}{M_u} - \overline{Z}(s't'u^{-2}) \frac{M_u(s't'u^{-2})}{M_u} \right) &= |st - s't'| \\
 (3.7) \quad &= 2 \text{Var} \left(\frac{1}{\sqrt{2}} B(st) - \frac{1}{\sqrt{2}} B(s't) \right).
 \end{aligned}$$

For some small $\theta \in (0, 1)$, by (3.5) we obtain that for $t, t' \in [0, \theta]$

$$(3.8) \quad \mathbb{E}(\bar{Z}(t) - \bar{Z}(t'))^2 = 2 - 2r(t, t') \leq \mathbb{C}_1 |t - t'|$$

holds. By (3.4), (3.6), (3.7), (3.8) and Lemma 5.1 in [8], as $u \rightarrow \infty$,

$$\Pi_0^{+\varepsilon}(u) \sim \mathbb{E} \left\{ \sup_{t \in [0, (1+\varepsilon_1)(c^2/\delta^2 + \lambda)]} \inf_{s \in [(1+\varepsilon_2)a, 1]} \exp \left(\frac{\sqrt{2\delta}}{\sigma} B(st) - \frac{\delta}{\sigma^2} st - \frac{\delta}{\sigma^2} \left(\sqrt{st} - \frac{c}{\delta} \right)^2 \right) \right\} \Psi \left(\frac{u}{M_u} \right),$$

and

$$\Pi_0^{-\varepsilon}(u) \sim \mathbb{E} \left\{ \sup_{t \in [0, (1-\varepsilon_1)(c^2/\delta^2 + \lambda)]} \inf_{s \in [(1-\varepsilon_2)a, 1]} \exp \left(\frac{\sqrt{2\delta}}{\sigma} B(st) - \frac{\delta}{\sigma^2} st - \frac{\delta}{\sigma^2} \left(\sqrt{st} - \frac{c}{\delta} \right)^2 \right) \right\} \Psi \left(\frac{u}{M_u} \right).$$

Letting $\varepsilon_1, \varepsilon_2 \rightarrow 0$, we have

$$(3.9) \quad \Pi_0(u) \sim \mathbb{E} \left\{ \sup_{t \in [0, c^2/\delta^2 + \lambda]} \inf_{s \in [a, 1]} \exp \left(\frac{\sqrt{2\delta}}{\sigma} B(st) - \frac{\delta}{\sigma^2} st - \frac{\delta}{\sigma^2} \left(\sqrt{st} - \frac{c}{\delta} \right)^2 \right) \right\} \Psi \left(\frac{u}{M_u} \right), \quad u \rightarrow \infty.$$

Next we show that

$$\Pi_1(u) = o(\Pi_0(u)), \quad \Pi_2(u) = o(\Pi_0(u)), \quad \text{and} \quad \Pi_3(u) = o(\Pi_0(u)).$$

Let $Y(t), t \in \mathbb{R}$ be a stationary Gaussian process with continuous trajectories, unit variance and correlation function satisfying for a constant $\varepsilon_3 \in (0, \frac{1}{2})$

$$r_Y(t) = 1 - \frac{(1 + \varepsilon_3)}{2} |t|.$$

By (3.4) and Slepian inequality in [18], we have

$$\begin{aligned} \Pi_1(u) &\leq \sum_{i=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in (t_u + I_u(k))} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln t} e^{-\delta v} dB(v) - \frac{c}{\delta} (1 - t^{\frac{1}{2}}) \right) > u \right\} \\ &\leq \sum_{i=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in (t_u + I_u(k))} \bar{Z}(t) > \mathcal{A}_u(k) \right\} \\ &\leq \sum_{i=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in (t_u + I_u(k))} Y(t) > \mathcal{A}_u(k) \right\} \\ &= \sum_{i=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} Y(u^{-2}t) > \mathcal{A}_u(k) \right\} \end{aligned}$$

where $\mathcal{A}_u(k) := \frac{u}{M_u} \left(1 + \frac{1-\varepsilon_4}{2u^2} (\sqrt{u^2 t_u + k\lambda} - c/\delta)^2 - \frac{\varepsilon_4}{u^2} \right)$ and $\varepsilon_4 \in (0, 1)$ is a small constant. We observe that

$$(3.10) \quad \inf_{1 \leq k \leq N(u)} \mathcal{A}_u(k) \geq \frac{u}{M_u} \rightarrow \infty, \quad u \rightarrow \infty.$$

Further,

$$\begin{aligned} &\lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| \mathcal{A}_u^2(k) \frac{\text{Var}(Y(u^{-2}t_1) - Y(u^{-2}t_2))}{\frac{2\delta(1+\varepsilon_3)}{\sigma^2} |t_1 - t_2|} - 1 \right| \\ &= \lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| \mathcal{A}_u^2(k) \frac{2 - 2r_Y(u^{-2}t_1 - u^{-2}t_2)}{\frac{2\delta(1+\varepsilon_3)}{\sigma^2} |t_1 - t_2|} - 1 \right| \\ (3.11) \quad &= 0, \end{aligned}$$

and

$$\sup_{1 \leq k \leq N(u)} \sup_{\substack{|t_1 - t_2| < \epsilon \\ t_1, t_2 \in [0, \lambda]}} \mathcal{A}_u^2(k) \mathbb{E} \{ (Y(u^{-2}t_1) - Y(u^{-2}t_2)) Y(0) \}$$

$$\begin{aligned}
& \leq \mathbb{C}_2 u^2 \sup_{\substack{|t_1-t_2|<\epsilon \\ t_1, t_2 \in [0, \lambda]}} |r_Y(u^{-2}t_1) - r_Y(u^{-2}t_2)| \\
& \leq \mathbb{C}_3 u^2 \sup_{\substack{|t_1-t_2|<\epsilon \\ t_1, t_2 \in [0, \lambda]}} \left| \frac{1+\varepsilon_3}{2} u^{-2}(t_1 - t_2) \right| \\
(3.12) \quad & \leq \mathbb{C}_4 \sup_{\substack{|t_1-t_2|<\epsilon \\ t_1, t_2 \in [0, \lambda]}} |t_1 - t_2| \rightarrow 0, \quad u \rightarrow \infty, \quad \epsilon \rightarrow 0.
\end{aligned}$$

According to (3.10), (3.11), (3.12) and Lemma 5.3 of [5], we have as $u \rightarrow \infty, \varepsilon_4 \rightarrow 0, \lambda \rightarrow \infty$

$$\begin{aligned}
\Pi_1(u) & \leq \mathbb{C}_5 \lambda \sum_{k=1}^{N(u)} \Psi(\mathcal{A}_u(k)) \\
& \sim \mathbb{C}_5 \lambda \sum_{k=1}^{N(u)} \frac{1}{\sqrt{2\pi} \mathcal{A}_u(k)} e^{-\frac{\mathcal{A}_u^2(k)}{2}} \\
& \leq \mathbb{C}_5 \lambda \sum_{k=1}^{N(u)} \frac{M_u}{\sqrt{2\pi} u} \exp\left(-\frac{u^2}{2M_u^2} \left(1 + \frac{1-\varepsilon_4}{u^2} \left(\sqrt{u^2 t_u + k\lambda} - c/\delta\right)^2 - \frac{2\varepsilon_4}{u^2}\right)\right) \\
& \sim \mathbb{C}_5 \lambda \Psi\left(\frac{u}{M_u}\right) e^{\frac{\varepsilon_4}{M_u^2}} \sum_{k=1}^{N(u)} \exp\left(-\frac{1-\varepsilon_4}{2M_u^2} \left(\sqrt{u^2 t_u + k\lambda} - c/\delta\right)^2\right) \\
(3.13) \quad & \leq \mathbb{C}_6 \Psi\left(\frac{u}{M_u}\right) e^{\frac{\sigma^2 \varepsilon_4}{2\delta}} \lambda \sum_{k=1}^{\infty} e^{-\mathbb{C}_7 k\lambda} = o\left(\Psi\left(\frac{u}{M_u}\right)\right).
\end{aligned}$$

Moreover, for all u large

$$\begin{aligned}
\frac{1}{M_u(t)} - \frac{1}{M_u} & \geq \frac{[G_u(t)V_Z(t_u)]^2 - [G_u(t_u)V_Z(t)]^2}{2uV_Z^3(t_u)G_u(t_u)} \\
& = \frac{\frac{\sigma^2}{2\delta}(u^2 + \frac{2c}{\delta}u)(\sqrt{t} - \sqrt{t_u})^2}{2u[\frac{\sigma^2}{2\delta}(1-t_u)]^{3/2}[u + \frac{c}{\delta}(1-\sqrt{t_u})]} \\
& \geq \mathbb{C}_8(\sqrt{t} - \sqrt{t_u})^2 \\
& \geq \frac{\mathbb{C}_8 \left(\frac{\ln u}{u}\right)^4}{\left(\sqrt{\left(\frac{\ln u}{u}\right)^2 + \left(\frac{c}{\delta u + c}\right)^2} + \frac{c}{\delta u + c}\right)^2} \\
(3.14) \quad & \geq \mathbb{C}_8 \frac{(\ln u)^2}{u^2}
\end{aligned}$$

holds for any $t \in [t_u + \delta(u), \theta]$, therefore

$$\sup_{t \in [t_u + \delta(u), \theta]} M_u(t) \leq \left(\frac{1}{M_u} + \mathbb{C}_8 \frac{(\ln u)^2}{u^2} \right)^{-1}.$$

Thus the above inequality combined with (3.8) and Theorem 8.1 in [18] derives that

$$\begin{aligned}
\Pi_2(u) & \leq \mathbb{P} \left\{ \sup_{t \in [t_u + \delta(u), \theta]} \overline{Z}(t) M_u(t) > u \right\} \\
& \leq \mathbb{P} \left\{ \sup_{t \in [0, \theta]} \overline{Z}(t) > u \left(\frac{1}{M_u} + \mathbb{C}_8 \frac{(\ln u)^2}{u^2} \right) \right\} \\
& \leq \mathbb{C}_9 u^2 \Psi \left(u \left(\frac{1}{M_u} + \mathbb{C}_8 \frac{(\ln u)^2}{u^2} \right) \right) \\
(3.15) \quad & \leq o \left(\Psi \left(\frac{u}{M_u} \right) \right), \quad u \rightarrow \infty.
\end{aligned}$$

Finally, since

$$\sup_{t \in [\theta, 1]} V_Z^2(t) \leq \frac{\sigma^2}{2\delta}(1 - \theta), \quad \text{and} \quad \mathbb{E} \left\{ \sup_{t \in [\theta, 1]} Z(t) \right\} \leq \mathbb{C}_{10} < \infty,$$

by Borell inequality in [1]

$$(3.16) \quad \Pi_3(u) \leq \mathbb{P} \left\{ \sup_{t \in [\theta, 1]} Z(t) > u \right\} \leq \exp \left(-\frac{\delta(u - \mathbb{C}_{10})^2}{\sigma^2(1 - \theta)} \right) = o \left(\Psi \left(\frac{u}{M_u} \right) \right), \quad u \rightarrow \infty,$$

which combined with (3.2), (3.3), (3.9), (3.13) and (3.15) shows that

$$\psi(u) \sim \Pi_0(u), \quad u \rightarrow \infty.$$

Consequently, letting $\lambda \rightarrow \infty$, we have

$$\begin{aligned} \psi(u) &\sim \mathbb{E} \left\{ \sup_{t \in [0, \infty)} \inf_{s \in [a, 1]} \exp \left(\frac{\sqrt{2\delta}}{\sigma} B(st) - \frac{\delta}{\sigma^2} st - \frac{\delta}{\sigma^2} \left(\sqrt{st} - \frac{c}{\delta} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right) \\ &= \mathbb{E} \left\{ \sup_{t \in [0, \infty)} \inf_{s \in [a, 1]} \exp \left(\sqrt{2} B(st) - st - \left(\sqrt{st} - \frac{c}{\sigma\sqrt{\delta}} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right), \quad u \rightarrow \infty. \end{aligned}$$

□

PROOF OF THEOREM 2.3 We use the same notation as in the proof of Theorem 2.1. For $x \in \left(-\frac{c^2}{\delta^2}, \infty \right)$ and $u > 0$

$$\begin{aligned} &\mathbb{P} \left\{ u^2 \left(e^{-2\delta\eta(u)} - \left(\frac{c}{\delta u + c} \right)^2 \right) \leq x \mid \eta(u) < \infty \right\} \\ &= \frac{\mathbb{P} \left\{ \inf_{t \in [-\frac{1}{2\delta} \ln(t_u + u^{-2}x), \infty)} \sup_{s \in [t, t+T_u]} \tilde{R}_u^\delta(s) < 0 \right\}}{\mathbb{P} \left\{ \inf_{t \in [0, \infty)} \sup_{s \in [t, t+T_u]} \tilde{R}_u^\delta(s) < 0 \right\}} \\ &= \frac{\mathbb{P} \left\{ \sup_{t^* \in [0, t_u + u^{-2}x]} \inf_{s^* \in [t^* e^{-2\delta T_u}, t^*]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dB(v) - c \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dv \right) > u \right\}}{\mathbb{P} \left\{ \sup_{t^* \in [0, 1]} \inf_{s^* \in [t^* e^{-2\delta T_u}, t^*]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dB(v) - c \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dv \right) > u \right\}} \\ &= \mathbb{P} \left\{ u^2 (\tau_u^* - t_u) \leq x \mid \tau_u^* < 1 \right\}, \end{aligned}$$

where $\tau_u^* = \{t \geq 0 : \sigma \int_0^{-\frac{1}{2\delta} \ln t^*} e^{-\delta v} dB(v) - \frac{c}{\delta}(1 - t^{*\frac{1}{2}}) > u\}$.

For $\psi_x(u) := \mathbb{P} \left\{ \sup_{t^* \in [0, t_u + u^{-2}x]} \inf_{s^* \in [t^* e^{-2\delta T_u}, t^*]} \left(\sigma \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dB(v) - c \int_0^{-\frac{1}{2\delta} \ln s^*} e^{-\delta v} dv \right) > u \right\}$, using the similar argumentation about $\Pi_0(u)$ as in the proof of Theorem 2.1 with $\lambda = x$, we obtain

$$\begin{aligned} \psi_x(u) &\sim \mathbb{E} \left\{ \sup_{t \in [0, c^2/\delta^2 + x]} \inf_{s \in [a, 1]} \exp \left(\frac{\sqrt{2\delta}}{\sigma} B(st) - \frac{\delta}{\sigma^2} st - \frac{\delta}{\sigma^2} \left(\sqrt{st} - \frac{c}{\delta} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right) \\ &= \mathbb{E} \left\{ \sup_{t \in [0, \frac{c^2}{\sigma^2\delta} + \frac{\delta x}{\sigma^2}]} \inf_{s \in [a, 1]} \exp \left(\sqrt{2} B(st) - st - \left(\sqrt{st} - \frac{c}{\sigma\sqrt{\delta}} \right)^2 \right) \right\} \Psi \left(\frac{1}{\sigma} \sqrt{2\delta u^2 + 4cu} \right), \quad u \rightarrow \infty. \end{aligned}$$

Thus

$$\mathbb{P} \left\{ u^2 \left(e^{-2\delta\eta_u} - \left(\frac{c}{\delta u + c} \right)^2 \right) \leq x \mid \eta_u < \infty \right\} = \frac{\psi_x(u)}{\psi(u)} \sim \frac{\tilde{\mathcal{P}}_a^f[0, \frac{c^2}{\sigma^2\delta} + \frac{\delta x}{\sigma^2}]}{\tilde{\mathcal{P}}_a^f[0, \infty)}, \quad u \rightarrow \infty.$$

□

Acknowledgement: Thanks to Swiss National Science Foundation Grant no. 200021-166274.

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